OPEN STRINGS WITH TOPOLOGICALLY INSPIRED BOUNDARY CONDITIONS

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Abstract

We consider an open string described by an action of the Dirac-Nambu-Goto type with topological corrections which affect the boundary conditions but not the equations of motion. The most general addition of this kind is a sum of the Gauss-Bonnet action and the first Chern number (when the background spacetime dimension is four) of the normal bundle to the string worldsheet. We examine the modification introduced by such terms in the boundary conditions at the ends of the string.

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The simplest topological action for an open string which is dependent only on the intrinsic geometry is proportional to the lorentzian version of the Euler characteristic, χ. Ignoring boundary discontinuities, this is given by

$$4\pi\chi = \int_m d^2\xi\sqrt{-\gamma}\,\mathcal{R} + 2\int_{\partial m} d\tau\, k\,. \tag{1}$$

The bulk Gauss-Bonnet action must be supplemented with a boundary term to yield a topological invariant. Here τ is the proper time, and k the geodesic curvature along the end worldline. Thus the introduction of the bulk Gauss-Bonnet type term into the action for an open string induces a load on the ends, and it is equivalent to attaching "rigid" particles at the ends. The effect of adding this term to the Dirac-Nambu-Goto action is to alter the boundary conditions at the ends, leaving the equations of motion unaltered. This finds an application in the "stringy" description of hadronic physics [1, 2, 3].

At the same order, for a string living in a four-dimensional background spacetime, there is an additional topological invariant, constructed from the extrinsic geometry of the string worldsheet, given by the first Chern number of the normal bundle of the worldsheet. This is related to the self-intersection number of the worldsheet. The addition of this term is motivated by the construction of a string analogue of the θ-term in QCD [2, 4, 5, 6].

In this note, we present a simple derivation of the appropriate boundary conditions for a Dirac-Nambu-Goto open string supplemented with such topologically inspired terms in the action. We use simple variational techniques, employing the general results about the variation of the intrinsic and extrinsic geometry of a membrane of arbitrary dimension in an arbitrary background previously derived in Ref. [7]. With respect to previous treatments [8], our approach has the advantage of bringing to the forefront the geometrical content of the boundary conditions. Moreover, we can easily derive the appropriate boundary conditions when the background geometry is arbitrary. For additional recent studies on this topic, see Refs. [9, 10, 11, 12].

The system is defined by the action functional

$$S[X] = -\mu I_0 - \alpha I_1 - \beta I_2\,. \tag{2}$$

Here the field variable is X, the embedding functions of the worldsheet m swept out by the string in spacetime, defined by $x^\mu = X^\mu(\xi^a)$, where x^μ are local coordinates in the background spacetime ($\mu, \nu = 0, 1, 2, 3$), and ξ^a are coordinates on the worldsheet m ($a, b = 0, 1$); μ is the tension in the string, α and β are two dimensionless numerical parameters.

The first term is proportional to the area of the worldsheet m, and is known as the Dirac-Nambu-Goto action, with the area defined by

$$I_0 = \int_m d^2\xi\sqrt{-\gamma} \tag{3}$$

where γ denotes the determinant of the metric induced on m by the background spacetime metric,

$$\gamma_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \,. \tag{4}$$

The second term is proportional to the bulk Gauss-Bonnet action,

$$I_1 = \int_m d^2\xi \sqrt{-\gamma}\,\mathcal{R}\,, \tag{5}$$

where $\mathcal{R}$ is the scalar curvature constructed with the induced metric on the string worldsheet.

The last term is defined by

$$I_2 = \int_m d^2\xi \,\widetilde{\Omega}\,. \tag{6}$$

The scalar density $\widetilde{\Omega}$ is constructed from the extrinsic twist curvature of the extrinsic twist potential, ${\omega_a}^{ij}$, the connection associated with the freedom of normal rotations (see *e.g.* [7])

$${\Omega_{ab}}^{ij} = \partial_b {\omega_a}^{ij} - \partial_a {\omega_b}^{ij}\,, \tag{7}$$

by contracting the normal $(i,j = 1,2)$ and worldsheet index pairs with the corresponding Levi-Civita antisymmetric symbols:

$$\widetilde{\Omega} := \frac{1}{2}\epsilon_{ij}\epsilon^{ab}{\Omega_{ab}}^{ij}\,. \tag{8}$$

It is now obvious that $\widetilde{\Omega}$ is a total divergence so that I_2 is a topological invariant. In fact, it is proportional to the first Chern number of the normal bundle of m. We note that the tensor density ϵ^{ab} (assuming the values 0 and ± 1) is related to the tensor ε^{ab} by $\epsilon^{ab} = \varepsilon^{ab}\sqrt{-\gamma}$.

The curvatures $\mathcal{R}_{abcd}$ and ${\Omega_{ab}}^{ij}$ are related to the extrinsic and background geometry, as characterized by the extrinsic curvature ${K_{ab}}^i$ and the spacetime Riemann tensor ${R^\mu}_{\nu\alpha\beta}$ by the following integrability conditions, the Gauss-Codazzi and the Ricci equations, respectively:

$$\mathcal{R}_{abcd} = {K_{ac}}^i K_{bd\,i} - {K_{ad}}^i K_{bc\,i} + R_{\mu\nu\alpha\beta}\, e^\mu_a e^\nu_b e^\alpha_c e^\beta_d\,, \tag{9}$$

and

$${\Omega_{ab}}^{ij} = {K_{ac}}^i {K_b}^{c\,j} - {K_{bc}}^i {K_a}^{c\,j} + R_{\mu\nu\alpha\beta}\, e^\mu_a e^\nu_b n^{i\,\alpha} n^{j\,\beta}\,, \tag{10}$$

where ${e^\mu}_a$ denote the two vectors tangent to the worldsheet, and $n^{\mu\,i}$ the two unit vectors normal to the worldsheet.

There is an additional constraint on the covariant derivative of the extrinsic curvature given by the the Codazzi-Mainardi equations:

$$\widetilde{\nabla}_a {K_{bc}}^i - \widetilde{\nabla}_b {K_{ac}}^i = R_{\mu\nu\alpha\beta} e^\mu_a e^\nu_b e^\alpha_c n^{i\,\beta}\,. \tag{11}$$

Here $\widetilde{\nabla}_a$ denotes the $O(2)$ worldsheet covariant derivative

$$\widetilde{\nabla}_a = \nabla_a - \omega_a \,. \tag{12}$$

In particular, if the background geometry has constant curvature so that

$$R_{\mu\nu\alpha\beta} = \frac{R}{12}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) \,, \tag{13}$$

the projections appearing in both Eqs.(10) and (11) vanish. It follows that $\Omega^{ij}{}_{ab}$ is completely constructed out of the extrinsic curvature, and it assumes the simple form:

$$\widetilde{\Omega} = \epsilon_{ij}\epsilon^{ab}K_{ac}{}^{i}K_{bd}{}^{j}\gamma^{cd} \,. \tag{14}$$

The price one pays for this simple form, unfortunately, is that it is no longer obvious that $\widetilde{\Omega}$ is related to a curvature.

When the background spacetime M is flat, the worldsheet scalar curvature can also be expressed in terms of the extrinsic curvature, and one has that

$$\mathcal{R} = K^i K_i - K_{ab}{}^{i}K^{ab}{}_{i} \,, \tag{15}$$

where $K^i = \gamma^{ab}K_{ab}{}^{i}$ is the mean extrinsic curvature.

We are interested in the variation of the functional $S[X]$ with respect to an infinitesimal variation of the embedding functions

$$X^\mu \to X^\mu + \delta X^\mu \,. \tag{16}$$

The variation can be decomposed into a part normal and a part tangential to the worldsheet,

$$\begin{aligned} \delta X^\mu &= \delta_\perp X^\mu + \delta_\| X^\mu \\ &= \Phi_i n^{\mu\, i} + \Phi^a e^\mu{}_a \,. \end{aligned} \tag{17}$$

For the variation of the Dirac-Nambu-Goto action, all that is needed is the well known variation of the intrinsic metric,

$$\delta\gamma_{ab} = 2K_{ab}{}^{i}\Phi_i + 2\nabla_{(a}\Phi_{b)} \,, \tag{18}$$

where the second term is the worldsheet Lie derivative of γ_{ab} along the worldsheet vector Φ^a. Using this expression, it is a simple exercise to obtain the variation of the worldsheet area in the form,

$$\begin{aligned} \delta I_0 &= \int_m d^2\xi\sqrt{-\gamma}\,[K^i\Phi_i + \nabla_a\Phi^a] \\ &= \int_m d^2\xi\sqrt{-\gamma}K^i\Phi_i + \int_{\partial m} d\tau\,\eta_a\Phi^a \,, \end{aligned} \tag{19}$$

where we have used Stoke's theorem; τ is the proper time induced on the string end by γ_{ab}, and η^a is the space-like unit normal to the boundary as embedded in the worldsheet.

The variation of the two topological terms by their nature (we will show this explicitly) only contributes in boundary terms. Only the Dirac-Nambu-Goto term makes a bulk contribution under a normal deformation of the string worldsheet. Therefore, the Euler-Lagrange equations for our system are

$$K^i = 0\,, \tag{20}$$

the mean extrinsic curvature vanishes. We comment below on the additional boundary term in Eq. (19).

For the variations of the two topological terms it is useful to consider separately their tangential and normal variations. For the former, we do not need to know anything about the specific form of the Lagrangian. For any Lagrangian $L[X]$, since a tangential variation is just a reparameterization of the worldsheet,we have that

$$\delta_{\|} \int_m d^2\xi \sqrt{-\gamma}\, L = \int_{\partial m} d\tau L\, \eta^a\, \Phi_a\,. \tag{21}$$

Hence

$$\delta_{\|} S[X] = - \int_{\partial m} d\tau\, (\mu + \alpha \mathcal{R} + \beta \Omega)\, \eta^a \Phi_a\,, \tag{22}$$

where we have defined Ω with $\widetilde{\Omega} = \Omega \sqrt{-\gamma}$. Stationarity of the action under tangential deformations requires that the Lagrangian must vanish on ∂m,

$$\mu + \alpha \mathcal{R} + \beta \Omega = 0\,. \tag{23}$$

This is a first boundary condition to be implemented in addition to the equations of motion (20).

It may appear that an alternative possibility is to have the normal η^a go null at the ends, as in the standard boundary conditions for a Dirac-Nambu-Goto open string, making the ends move at the speed of light. However, since the introduction of I_1 implies a load on the ends, as long as $\alpha \neq 0$, generally these will follow a timelike worldline, except at isolated points such as cusps.

Let us turn now to the normal variation of the topological terms. For the sake of simplicity, we first assume that the background spacetime is flat. We will consider later an alternative strategy when the background spacetime is arbitrary.

To obtain the normal variation of I_1, we use Eq. (15) to express the scalar curvature $\mathcal{R}$ in terms of the extrinsic curvature. Under a normal displacement of the worldsheet, when the background is flat, the extrinsic curvature varies according to [7]:

$$\delta_{\perp} K_{ab}{}^i = - \widetilde{\nabla}_a \widetilde{\nabla}_b \Phi^i + K_{ac}{}^i K_b{}^{c\,j} \Phi_j\,. \tag{24}$$

Using this expression, together with Eq. (18), one finds that

$$
\begin{aligned}
\delta_\perp I_1 = & \;\; 2 \int_m d^2\xi \sqrt{-\gamma} \left(K^{ab}{}_i - K_i \gamma^{ab} \right) \widetilde{\nabla}_a \widetilde{\nabla}_b \Phi^i \\
= - & \;\; 2 \int_m d^2\xi \sqrt{-\gamma} \left[\widetilde{\nabla}_a \left(K^{ab}{}_i - \gamma^{ab} K_i \right) \right] \widetilde{\nabla}_b \Phi_i \\
+ & \;\; 2 \int_{\partial m} d\tau \eta_a \left(K^{ab}{}_i - K_i \gamma^{ab} \right) \widetilde{\nabla}_b \Phi^i \, .
\end{aligned} \tag{25}
$$

In the first line, there is a cancellation of terms that do not involve derivatives of Φ^i which makes use of the once-contracted Gauss-Codazzi equation $\mathcal{R}_{ab} = K_i K_{ab}{}^i - K_{ac}{}^i K_b{}^c{}_i$ and the two-dimensional fact $\mathcal{R}_{ab} = (1/2)\mathcal{R}\gamma_{ab}$. In the second line, we have integrated by parts, and we have used Stoke's theorem in the second term. The first integral over the worldsheet vanishes, because of the contracted Codazzi-Mainardi equations (11).

We need now to isolate the two independent variations on the boundary, Φ^i and its derivative along the normal to the boundary onto the worldsheet. To do this, it is convenient to exploit the completeness of the orthonormal basis $\{\eta^a, v^a\}$ on the boundary, with η^a the unit vector normal to the boundary into m, and v^a the unit vector tangent to the boundary. Then, at the timelike boundary, we can write the completeness relation

$$
\gamma_{ab} = \eta_a \eta_b - v_a v_b \, , \tag{26}
$$

and we can decompose the worldsheet covariant derivative into parts tangential and normal to the boundary as

$$
\widetilde{\nabla}_a = \eta_a (\eta^b \widetilde{\nabla}_b) - v_a (v^b \widetilde{\nabla}_b) = \eta_a \widetilde{\nabla}_{(\eta)} - v_a \widetilde{\mathcal{D}} \, . \tag{27}
$$

Here $\widetilde{\mathcal{D}} := v^a \widetilde{\nabla}_a = \partial_\tau - v^a \omega_a^{ij}$.

Using these expressions in Eq. (25) we have that

$$
\delta_\perp I_1 = 2 \int_{\partial m} d\tau \left[v^a v^b K_{ab}{}^i (\widetilde{\nabla}_{(\eta)} \Phi_i) - \eta^a v^b K_{ab}{}^i (\widetilde{\mathcal{D}} \Phi_i) \right] \, . \tag{28}
$$

We integrate by parts on the boundary to remove the derivative from $\widetilde{\mathcal{D}}\Phi_i$, and we define the projections of the extrinsic curvature at the boundary by

$$
\begin{aligned}
K_\parallel{}^i &= K_{ab}{}^i v^a v^b \, , \\
K_{\parallel\perp}{}^i &= K_{ab}{}^i v^a \eta^b \, , \\
K_\perp{}^i &= K_{ab}{}^i \eta^a \eta^b \, ,
\end{aligned} \tag{29}
$$

so that, finally, we obtain

$$
\delta_\perp I_1 = 2 \int_{\partial m} d\tau \left[K_{\parallel\, i} \left(\widetilde{\nabla}_{(\eta)} \Phi_i \right) + \left(\widetilde{\mathcal{D}} K_{\perp\parallel}{}^i \right) \Phi_i \right] \, . \tag{30}
$$

Now, we evaluate the normal variation of I_2, again assuming that the background is flat. We can use Eq.(14) to express $\widetilde{\Omega}$ in terms of the extrinsic curvature, so that

$$\begin{aligned}\delta_\perp I_2 &= 2\int_m d^2\xi\, \epsilon_{ij}\epsilon^{ab}[K_a{}^{d\,i}(\delta_\perp K_{bd}{}^j) + K_{ac}{}^i K^a{}_d{}^j(\delta_\perp \gamma^{cd})] \\ &= -2\int_m d^2\xi\, \epsilon_{ij}\epsilon^{ab} K_a{}^{c\,i}\widetilde{\nabla}_b\widetilde{\nabla}_c\Phi^j\,, \end{aligned} \tag{31}$$

where we have used Eqs. (18) and (24).

We now apply Stoke's theorem, to find

$$\delta_\perp I_2 = 2\int_m d^2\xi\, \epsilon_{ij}\epsilon^{ab}\widetilde{\nabla}_b K_{ac}{}^i\widetilde{\nabla}^c\Phi^j - 2\int_{\partial m} d\tau\, \epsilon_{ij}\varepsilon^{ab}\eta_b K_{ac}{}^i\widetilde{\nabla}^c\Phi^j\,. \tag{32}$$

The integral over m vanishes. To see this, we again exploit the Codazzi-Mainardi equations, Eq.(11), the right hand side of which is manifestly symmetric in a and b. This term therefore vanishes on contraction with ε^{ab}. Finally, we again exploit Eq.(27) and Stoke's theorem on the end worldline to express $\delta_\perp I_2$ as a sum of two independent variations:

$$\delta_\perp I_2 = -2\int_{\partial m} d\tau \epsilon_{ij}\varepsilon^{ab}\left[\widetilde{\mathcal{D}}(\eta_b K_{ac}{}^i v^c)\,\Phi^j + \eta_b\eta^c K_{ac}{}^i\,\widetilde{\nabla}_{(\eta)}\Phi^j\right]\,.. \tag{33}$$

In the sequel it will be useful to exploit the identity

$$\varepsilon^{ab} = v^a\eta^b - \eta^a v^b\,. \tag{34}$$

We note that

$$\varepsilon^{ab}v_b = \eta^a, \qquad \varepsilon^{ab}\eta_b = v^a\,. \tag{35}$$

For the normal variation of I_2, we obtain

$$\delta_\perp I_2 = 2\int_{\partial m} d\tau \epsilon_{ij}\left[K_{\perp\|}{}^j(\widetilde{\nabla}_{(\eta)}\Phi^i) + (\widetilde{\mathcal{D}}K_\|{}^j)\,\Phi^i\right]\,. \tag{36}$$

As expected, the normal variation of the topologically inspired terms gives only boundary terms involving some projections of the extrinsic curvature along the boundary , together with its first derivative along the boundary.

It is now possible to write down the remaining boundary conditions, in addition to Eq. (23), for the system defined by (2) as

$$\widetilde{\mathcal{D}}\left[\alpha K_{\|\perp}{}^i + \beta\epsilon^{ij}K_{\|\,j}\right] = 0\,, \tag{37}$$

and

$$\alpha K_\|{}^i + \beta\epsilon^{ij}K_{\|\perp\,j} = 0\,. \tag{38}$$

The boundary conditions, Eqs. (23), (37), and (38), supplement the bulk equations of motion, Eq. (20). Eqs. (23) and (38) are of second order in time derivatives of the ends embedding functions, whereas Eq. (37) is of third order.

We can now use the boundary conditions to evaluate the curvatures at the boundary. At the boundary, the curvatures are given by

$$\begin{aligned} \mathcal{R} &= 2(K_{\perp\parallel}{}^i K_{\perp\parallel\, i} - K_{\parallel}{}^i K_{\perp\, i})\,, && (39)\\ \Omega &= 2\epsilon_{ij} K_{\perp\parallel}{}^i (K_{\parallel}{}^j + K_{\perp}{}^j)\,. && (40)\end{aligned}$$

When m is extremal, $K_\perp{}^i = K_\parallel{}^i$. This gives

$$\begin{aligned} \mathcal{R} &= 2(K_{\perp\parallel}{}^i K_{\perp\parallel\, i} - K_{\parallel}{}^i K_{\parallel\, i})\,, && (41)\\ \Omega &= 4\epsilon_{ij} K_{\perp\parallel}{}^i K_{\parallel}{}^j\,. && (42)\end{aligned}$$

On the boundary, using this together with (38), one obtains

$$\begin{aligned} \mathcal{R} &= \frac{2}{\beta^2}(\alpha^2 - \beta^2) K_\parallel{}^i K_{\parallel\, i}\,, && (43)\\ \Omega &= 4\frac{\alpha}{\beta} K_\parallel{}^i K_{\parallel\, i}\,. && (44)\end{aligned}$$

The boundary condition (23) then fixes the squared norm, $K_\parallel{}^i K_{\parallel\, i}$, completely in terms of the three parameters, μ, α and β:

$$K_\parallel{}^i K_{\parallel\, i} = \frac{\mu\beta^2}{2\alpha}\frac{1}{\alpha^2+\beta^2}\,. \qquad (45)$$

This in turn fixes the boundary values of the curvatures:

$$\begin{aligned} \mathcal{R} &= \frac{\mu}{\alpha}\left(\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}\right) \\ \Omega &= \frac{2\mu\beta}{\alpha^2+\beta^2}\,. && (46)\end{aligned}$$

We note, in particular, that when $\beta \neq 0$, $\alpha > 0$ is necessary for the consistency of Eq.(45). β may, however, assume either sign.

We have seen that the norm of $K_\parallel{}^i$ is fixed completely on the ends by the parameters appearing in the action, μ, α and β. The remaining boundary condition Eq.(37) modulo (38) implies that

$$\widetilde{\mathcal{D}} K_\parallel{}^i = 0\,. \qquad (47)$$

$K_\parallel{}^i$ is covariantly constant along the end. Note that this is consistent with the fact that the norm is fixed. The freedom is to choose the ratio, $K_\parallel{}^2/K_\parallel{}^1$. This we input as an initial condition.

Of course, the worldsheet cannot be totally geodesic, $K_{ab}{}^i = 0$, since this would imply $\mu = 0$. Moreover, no solutions to our system exist in which the string moves on a plane, such that the extrinsic curvature along one normal is vanishing, say $K_{ab}{}^1 = 0$, for the same reason. In order to allow for planar solutions, we must set $\beta = 0$.

These boundary conditions may also be seen in terms of the geometric quantities associated with the direct embedding of the ends worldlines in spacetime [13]. Then $K_\parallel{}^i$ coincides with the extrinsic curvature of the end worldline along the unit normal vectors $n^{\mu\, i}$. The extrinsic twist potential of the ends is made up by $K_{\parallel\perp}{}^i$, which is the mixed part along $n^{\mu\, i}$ and η^μ, and by $v^a \omega_a{}^{ij}$. The only geometric quantity of the ends that does not appear in the boundary conditions is k, the extrinsic curvature along η^a.

It is interesting to consider various reduction of the system defined by S. When $\alpha = 0$, we can no longer assume that the trajectory will be timelike, and (23) is substituted by the requirement that the ends go null. The boundary condition (38) then implies that Ω vanishes, so that the extrinsic twist potential is pure gauge at the ends. Also when $\beta = 0$, we have that Ω vanishes at the ends; moreover, now $\mathcal{R}$ is proportional to the norm of $K_{\perp\parallel}{}^i$, and it is positive definite.

Let us suppose that in addition massive particles, of mass M, are attached to the ends. Using the results of Refs. [14, 15], we have that the boundary conditions (23) and (38) are modified respectively to

$$\mu + \alpha \mathcal{R} + \beta \Omega + M k = 0\,, \tag{48}$$

where k denotes the geodesic curvature of the ends into m, and

$$(\alpha + M/2) K_\parallel{}^i + \beta \epsilon^{ij} K_{\parallel\perp\, j} = 0\,. \tag{49}$$

Eq.(37) is unchanged. As before, $K_\parallel{}^i$ is covariantly constant which in turn implies that its norm is too. This means that k is also constant on the end. The end particle moves with constant acceleration exactly as it would without the topological modification.

We consider now the case in which the background spacetime geometry is left arbitrary. In this case there is no calculational advantage in employing the integrability conditions (9) and (10). We follow an alternative strategy, which also provides an independent check of the validity of the boundary conditions we have derived above.

To obtain the normal variation of I_1, we need to know how the scalar curvature varies. One has that,

$$\delta \mathcal{R}^a{}_{bcd} = \nabla_c (\delta \gamma_{db}{}^a) - \nabla_d (\delta \gamma_{cb}{}^a)\,, \tag{50}$$

where $\gamma_{ab}{}^c$ is the affine connection compatible with γ_{ab}. For the scalar curvature, $\mathcal{R} = \mathcal{R}^a{}_{bad}\gamma^{bd}$, we have

$$\delta_\perp \mathcal{R} = \nabla_c (\gamma^{ab} \delta_\perp \gamma_{ab}{}^c) - \nabla^b (\delta_\perp \gamma_{ab}{}^a) - 2 \mathcal{R}_{ab} K_i^{ab} \Phi^i\,, \tag{51}$$

where [7]

$$\delta_\perp \gamma_{ab}{}^c = \gamma^{cd}\left[\nabla_a (K_{bd}{}^i \Phi_i) + \nabla_b (K_{ad}{}^i \Phi_i) - \nabla_d (K_{ab}{}^i \Phi_i)\right] . \tag{52}$$

We therefore obtain

$$\delta_\perp I_1 = 2\int_{\partial m} d\tau \, \eta^b \left[\nabla_a (K^a{}_b{}^i \Phi_i) - \nabla_b (K^i \Phi_i)\right] , \tag{53}$$

where we have used the 2-dimensional identity

$$\mathcal{R}_{ab} = \frac{1}{2}\gamma_{ab}\mathcal{R} , \tag{54}$$

to reduce the variation to a total divergence, and Stoke's theorem to express it as an integral over the ends.

To express Eq. (53) in terms of the independent variations at the boundary, we use Eqs. (26) and (27), so that

$$\begin{aligned} \delta_\perp I_1 &= 2\int_{\partial m} d\tau \Big[\Big(\eta^a \eta^b \widetilde{\nabla}_{(\eta)} K_{ab}{}^i - \widetilde{\nabla}_{(\eta)} K^i - v^a \eta^b \widetilde{\mathcal{D}} K_{ab}{}^i + \widetilde{\mathcal{D}}(v^a \eta^b K_{ab}{}^i) \Big) \Phi_i \\ &\quad + (\eta^a \eta^b K_{ab}{}^i - K_i) \widetilde{\nabla}_{(\eta)} \Phi_i \Big] . \end{aligned} \tag{55}$$

We note that the sum of the first two terms gives

$$\begin{aligned} \eta^a \eta^b \widetilde{\nabla}_{(\eta)} K_{ab}{}^i - \widetilde{\nabla}_{(\eta)} K^i &= v^a v^b \eta^c \widetilde{\nabla}_c K_{ab}{}^i \\ &= v^a v^b \eta^c \widetilde{\nabla}_b K_{ac}{}^i + R_{\mu\nu\alpha\beta}\, v^\mu \eta^\nu v^\alpha n^{\beta\, i} \\ &= v^a \eta^b \widetilde{\mathcal{D}} K_{ab}{}^i + R_{\mu\nu\alpha\beta}\, v^\mu \eta^\nu v^\alpha n^{\beta\, i} \quad , \end{aligned} \tag{56}$$

where we have exploited the Codazzi-Mainardi equations (11) on the second line, and $v^\mu = e^\mu{}_a v^a \, , \eta^\mu = e^\mu{}_a \eta^a$. As a result, the first three terms on the right hand side of Eq.(55) get replaced by the projection of the spacetime Riemann tensor appearing in (56). Hence, we find that the normal variation of I_1 is now given by

$$\delta_\perp I_1 = 2\int_{\partial m} d\tau \left[K^i_\parallel (\widetilde{\nabla}_{(\eta)} \Phi_i) + (\widetilde{\mathcal{D}} K_{\parallel\perp}{}^i + R_{\mu\nu\alpha\beta}\, v^\mu \eta^\nu v^\alpha n^{\beta\, i}) \Phi_i \right] . \tag{57}$$

For the normal variation of I_2, we use the fact that the extrinsic twist curvature varies according to

$$\delta(\Omega_{ab}{}^{ij} \epsilon_{ij}) = \nabla_b (\delta\omega_a{}^{ij} \epsilon_{ij}) - \nabla_a (\delta\omega_b{}^{ij} \epsilon_{ij}) . \tag{58}$$

In particular, under a normal deformation of the string worldsheet [7],

$$\delta_\perp (\omega_a^{ij} \epsilon_{ij}) = -2K_{ab}{}^i \widetilde{\nabla}^b \Phi^j \epsilon_{ij} + R_{\mu\nu\alpha\beta}\, n^{\mu\, i} n^{\nu\, j} n^{\alpha\, k} e^\beta{}_a \Phi_k \epsilon_{ij} . \tag{59}$$

We have then

$$
\begin{aligned}
\delta_\perp I_2 &= \int_{\partial m} d\tau\, \epsilon_{ij}\varepsilon^{ab}\eta_b(-2K_{ac}{}^i\widetilde{\nabla}^c\Phi^j + R_{\mu\nu\alpha\beta}\, n^{\mu\, i}n^{\nu\, j}n^{\alpha\, k}e^\beta{}_b\Phi_k) \\
&= \int_{\partial m} d\tau\, \left[2(\widetilde{\mathcal{D}}K_\parallel^j)\epsilon_{ij}\Phi^i + R_{\mu\nu\alpha\beta}\, n^{\mu\, k}n^{\nu\, l}n^{\alpha\, i}v^\beta\Phi_i\epsilon_{kl} + 2K_{\perp\parallel}{}^j\epsilon_{ij}(\widetilde{\nabla}_{(\eta)}\Phi^i)\right] . \quad (60)
\end{aligned}
$$

Therefore we find that the boundary conditions (23) and (38) are not affected by the generalization to an arbitrary background spacetime, whereas (37) is modified to

$$
\widetilde{\mathcal{D}}\left[\alpha K_{\parallel\perp}{}^i + \alpha R_{\mu\nu\rho\sigma}v^\mu\eta^\nu v^\rho n^{\sigma\, i} + \beta\epsilon^{ij}K_{\parallel\, j} + \frac{\beta}{2}R_{\mu\nu\rho\sigma}n^{\mu\, k}n^{\nu\, l}n^{\rho\, i}v^\sigma\epsilon_{kl}\right] = 0\,, \qquad (61)
$$

In particular, note that the additional curvature terms vanish when the background spacetime has constant curvature.

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